

Pseudocyclic Association Schemes and Strongly Regular Graphs

Version of April 16, 2009

TAKUYA IKUTA¹

Graduate School of Information Sciences, Tohoku University, Aramaki-Aza-Aoba, Aoba-ku, Sendai, 980-8579 Japan

ikuta@law.kobegakuin.ac.jp

AKIHIRO MUNEMASA

Graduate School of Information Sciences, Tohoku University, Aramaki-Aza-Aoba, Aoba-ku, Sendai, 980-8579 Japan

munemasa@math.is.tohoku.ac.jp

Abstract. Let \mathcal{X} be a pseudocyclic association scheme in which all the nontrivial relations are strongly regular graphs with the same eigenvalues. We prove that the principal part of the first eigenmatrix of \mathcal{X} is a linear combination of an incidence matrix of a symmetric design and the all-ones matrix. Amorphous pseudocyclic association schemes are examples of such association schemes whose associated symmetric design is trivial. We present several non-amorphous examples, which are either cyclotomic association schemes, or their fusion schemes. Special properties of symmetric designs guarantee the existence of further fusions, and the two known non-amorphous association schemes of class 4 discovered by van Dam and by the authors, are recovered in this way. We also give another pseudocyclic non-amorphous association scheme of class 7 on $\text{GF}(2^{21})$, and a new pseudocyclic amorphous association scheme of class 5 on $\text{GF}(2^{12})$.

Keywords: association scheme, cyclotomy, finite field, strongly regular graph

1 Introduction

A.V. Ivanov's conjecture [13], though disproved by E.R. van Dam, asserted that, if each nontrivial relation in a symmetric association scheme is strongly regular, then an arbitrary partition of the set of nontrivial relations gives rise to an association scheme. Association schemes satisfying this conclusion is called *amorphous* (or *amorphic*). A counterexample to A.V. Ivanov's conjecture was given by van Dam in [7] for the imprimitive case, and in [8] for the primitive case. Presently there are only a few primitive counterexamples known. An example due to van Dam [8] has the first eigenmatrix given by

$$\begin{bmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & -15 & -15 & 17 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & 17 & -15 & -15 \end{bmatrix}, \quad (1)$$

¹permanent address: Faculty of Law, Kobe Gakuin University, Minatojima, Chuo-ku, Kobe, 650-8586 Japan

and another one is due to the authors [12] with the first eigenmatrix given by

$$\begin{bmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 204 & -239 & -239 & 273 \\ 1 & 204 & -239 & 273 & -239 \\ 1 & 204 & 273 & -239 & -239 \end{bmatrix}. \quad (2)$$

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is said to be pseudocyclic if the nontrivial multiplicities m_1, \dots, m_d of \mathcal{X} coincide. The first eigenmatrix of a pseudocyclic association scheme is of the form

$$P = \begin{bmatrix} 1 & f & \dots & f \\ 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{bmatrix},$$

where f denotes the common nontrivial multiplicities, as well as the common nontrivial valencies. The submatrix P_0 is called the principal part of P . If we restrict A.V. Ivanov's conjecture to the pseudocyclic case, it asserts that for pseudocyclic association scheme in which each nontrivial relation is strongly regular, the principal part of its first eigenmatrix is a linear combination of I and J , after a suitable permutation of rows.

Cyclotomic association schemes in which each nontrivial relation is strongly regular, have been investigated in its own right. It follows from McEliece's theorem ([17], see also [19, Lemma 2.8]) that the number of nontrivial eigenvalues of the cyclotomic association scheme of class e over $\text{GF}(p^m)$ is the same as that of weights in the irreducible cyclic code $c(p, m, e)$ (see [19, Definition 2.2]). In this sense, such cyclotomic association schemes correspond to two-weight irreducible cyclic codes. Moreover, under this correspondence, subfield codes, semiprimitive codes correspond to amorphous cyclotomic association scheme which are imprimitive, primitive, respectively. Primitive amorphous cyclotomic association schemes were investigated by Baumert, Mills and Ward [3], and Brouwer, Wilson and Xiang [6]. Non-amorphous cyclotomic association schemes in which every nontrivial relation is strongly regular, are thus equivalent to exceptional two weight irreducible cyclic codes in the sense of Schmidt and White [19]. Therefore, cyclotomic association schemes corresponding to exceptional two weight irreducible cyclic codes are pseudocyclic counterexamples to A.V. Ivanov's conjecture, and there are eleven such codes in [19].

One of the purpose of this paper is to show that both of the counterexamples with first eigenmatrices (1), (2) are derived from some pseudocyclic association schemes $\mathcal{X}_1, \mathcal{X}_2$, respectively, of class 15 which are also counterexamples themselves. It turns out that, the principal part of the first eigenmatrix of \mathcal{X}_1 or \mathcal{X}_2 is expressed by an incidence matrix of $\text{PG}(3, 2)$. In a more general setting, we prove in Theorem 3.1 that the principal part of the first eigenmatrix is a linear combination of an incidence matrix of a symmetric design and the all-ones matrix. For an amorphous pseudocyclic association scheme of class d , the associated symmetric design is the complete 2 -($d, d-1, d-2$) design. When the associated symmetric design is a projective space, we show in Theorem 4.1 that the existence of certain fusion schemes follows from special properties of projective spaces. This gives an explanation for the existence of the fusion schemes of class 4 in $\mathcal{X}_1, \mathcal{X}_2$.

Moreover, the two pseudocyclic association schemes $\mathcal{X}_1, \mathcal{X}_2$ of class 15 give rise to two pseudocyclic amorphous fusion schemes of class 5. We also give a pseudocyclic class 7 fusion scheme of the cyclotomic association scheme of class 49 on $\text{GF}(2^{21})$. Its associated design is $\text{PG}(2, 2)$.

2 Preliminaries

We refer the reader to [2] for notation and general theory of association schemes. Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class d on X . Let P be the first eigenmatrix of \mathcal{X} . Let $\{\Lambda_j\}_{j=0}^{d'}$ be a partition of $\{0, 1, \dots, d\}$ with $\Lambda_0 = \{0\}$, and we set $R_{\Lambda_j} = \bigcup_{\ell \in \Lambda_j} R_\ell$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ forms an association scheme, then we call $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ a *fusion* scheme of \mathcal{X} . If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ is an association scheme for any partition $\{\Lambda_j\}_{j=0}^{d'}$ of $\{0, 1, \dots, d\}$ with $\Lambda_0 = \{0\}$, then \mathcal{X} is called *amorphous*. We refer the reader to a recent article [9] for details on amorphous association schemes.

There is a simple criterion in terms of P for a given partition $\{\Lambda_j\}_{j=0}^{d'}$ to give rise to a fusion scheme (due to Bannai [1], Muzychuk [18]): There exists a partition $\{\Delta_i\}_{i=0}^{d'}$ of $\{0, 1, \dots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j) -block of the first eigenmatrix P has a constant row sum. The constant row sum turns out to be the (i, j) entry of the first eigenmatrix of the fusion scheme.

An association scheme \mathcal{X} of class d having the nontrivial multiplicities $m_1 = \dots = m_d$ is called *pseudocyclic*. It is known that, in a pseudocyclic association scheme \mathcal{X} , all the nontrivial valencies coincide (see [2, p.76], or [5, Proposition 2.2.7]). By the *principal part* of the first eigenmatrix, we mean the lower-right $d \times d$ submatrix of the first eigenmatrix.

Let q be a prime power and let e be a divisor of $q - 1$. Fix a primitive element α of the multiplicative group of the finite field $\text{GF}(q)$. Then $\langle \alpha^e \rangle$ is a subgroup of index e and its cosets are $\alpha^i \langle \alpha^e \rangle$ ($0 \leq i \leq e - 1$). We define $R_0 = \{(x, x) \mid x \in \text{GF}(q)\}$ and $R_i = \{(x, y) \mid x - y \in \alpha^i \langle \alpha^e \rangle, x, y \in \text{GF}(q)\}$ ($1 \leq i \leq e$). Then $(\text{GF}(q), \{R_i\}_{i=0}^e)$ forms an association scheme and is called the *cyclotomic association scheme*, or *cyclotomic scheme*, for short, of class e on $\text{GF}(q)$. A cyclotomic scheme is a pseudocyclic association scheme.

Suppose $q = p^m$, where p is a prime. The cyclotomic scheme of class e on $\text{GF}(q)$ is amorphous if and only if m is even and e divides $p^{m'} + 1$ for some divisor m' of $m/2$. This is essentially due to Baumert, Mills and Ward [3], but see also [6].

3 A symmetric design in the first eigenmatrix

Theorem 3.1. *Let $(X, \{R_i\}_{i=0}^d)$ be a pseudocyclic association scheme of class d . Assume that the graphs (X, R_i) ($i = 1, \dots, d$) are all strongly regular with the same eigenvalues. Then there exists a symmetric $2-(d, k, \lambda)$ design \mathcal{D} such that the principal part of the first eigenmatrix of \mathcal{X} is given by $rM + s(J - M)$, where M is an incidence matrix of \mathcal{D} , r and s are the nontrivial eigenvalues of the graphs (X, R_i) .*

Proof. By the assumption, the principal part P_0 can be expressed as $P_0 = rM + s(J - M)$ for some $(0, 1)$ -matrix M . Then by the orthogonality relations (see [2, Chapter II, (3.10)]), we find

$$P_0 J = -J, \quad fJ + P_0 P_0^T = |X|I,$$

where f denotes the common nontrivial multiplicities. The former implies

$$MJ = -\frac{sd+1}{r-s}J,$$

hence $k = -(sd+1)/(r-s)$ is a positive integer. The latter implies

$$MM^T = \frac{1}{(r-s)^2}(|X|I + (s^2d + 2s - f)J).$$

This implies that M is an incidence matrix of a symmetric design on d points with block size k . \square

The assumption that the eigenvalues of the strongly regular graphs appearing as the nontrivial relations are the same, seems redundant. We have verified that the conclusion of Theorem 3.1 holds without this assumption for $d \leq 4$.

Next we show the existence of further fusions. We denote by $\mathbf{1}_n$ the column vector of length n whose entries are all 1.

Corollary 3.2. *Under the same assumptions as in Theorem 3.1, \mathcal{X} has a fusion scheme of class 3 with the first eigenmatrix*

$$\begin{bmatrix} 1 & f & (k-1)f & (d-k)f \\ 1 & r & (k-1)r & (d-k)s \\ 1 & r & (\lambda-1)r + (k-\lambda)s & (k-\lambda)r + (d-2k+\lambda)s \\ 1 & s & \lambda r + (k-1-\lambda)s & (k-\lambda)r + (d-2k+\lambda)s \end{bmatrix}. \quad (3)$$

In particular, there exists a fusion scheme of class 2 with the first eigenmatrix

$$\begin{bmatrix} 1 & kf & (d-k)f \\ 1 & kr & (d-k)s \\ 1 & \lambda r + (k-\lambda)s & (k-\lambda)r + (d-2k+\lambda)s \end{bmatrix}. \quad (4)$$

Proof. Let M be an incidence matrix of the design \mathcal{D} , so that $P_0 = rM + s(J - M)$ holds. Without loss of generality, we may assume that the first k columns of M correspond to the set of points on a block B of \mathcal{D} , and that B is represented by the first row of M . Let F denote the $d \times 3$ matrix defined by

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{1}_{k-1} & 0 \\ 0 & 0 & \mathbf{1}_{d-k} \end{bmatrix}.$$

Then we have

$$MF = \begin{bmatrix} 1 & k-1 & 0 \\ \mathbf{1}_{k-1} & (\lambda-1)\mathbf{1}_{k-1} & (k-\lambda)\mathbf{1}_{k-1} \\ 0 & \lambda\mathbf{1}_{d-k} & (k-\lambda)\mathbf{1}_{d-k} \end{bmatrix}.$$

It follows that the matrix P_0F has 3 distinct rows, which are precisely those of the 3×3 lower-right submatrix of (3). By the Bannai–Muzychuk criterion, we obtain a fusion scheme of class 3 with the first eigenmatrix given by (3). Fusing the first two relations of this class 3 association scheme, we obtain a class 2 association scheme with the first eigenmatrix given by (4). \square

Amorphous pseudocyclic association schemes satisfy the conditions of Theorem 3.1. However, the symmetric design appearing in the principal part of the first eigenmatrix is the complete $2-(d, d-1, d-2)$ design. The conclusion of Corollary 3.2 is trivially true for amorphous association schemes. The nontrivial part of Corollary 3.2 is that it holds also for non-amorphous association schemes.

Examples of cyclotomic schemes satisfying the conditions of Theorem 3.1 have been investigated thoroughly by Schmidt and White [19], and some of the exceptional examples were already found by Langevin [15]. The smallest example in [19, Table 1] is the cyclotomic scheme of class 11 on $\text{GF}(3^5)$, which gives a unique symmetric $2-(11, 5, 2)$ design by Theorem 3.1. Its associated strongly regular graph is the coset graph of the ternary Golay code (see [4]), which was later recognized as the cyclotomic graph by van Lint and Schrijver [16] in 1981. In this sense, a counterexample to A.V. Ivanov's conjecture [13] could be considered known before the conjecture was announced in 1991. We note that the fusion schemes of this cyclotomic association scheme obtained by Corollary 3.2 were already pointed out by Delsarte [10, Example 2 on p.93], in 1973.

There are three more pseudocyclic association schemes satisfying the conditions of Theorem 3.1, which are not cyclotomic schemes, but fusions of cyclotomic schemes. They will be given in the next section.

4 Projective spaces and fusion schemes

Let q be a prime power, m an integer greater than 1. By $\text{PG}(m, q)$ we mean the symmetric $2-(d, k, \lambda)$ design consisting of the points and hyperplanes of the projective space $\text{PG}(m, q)$ of dimension m over $\text{GF}(q)$, where $d = (q^{m+1} - 1)/(q - 1)$, $k = (q^m - 1)/(q - 1)$, $\lambda = (q^{m-1} - 1)/(q - 1)$. Let M be the hyperplane-point incidence matrix of $\text{PG}(m, q)$, and suppose that the columns of M are indexed by the points of $\text{PG}(m, q)$ in such a way that the last $q + 1$ columns correspond to the set of points on a line $L = \{\beta_1, \dots, \beta_{q+1}\}$. Consider the following $d \times (q + 2)$ matrix

$$F_1 = \begin{bmatrix} \mathbf{1}_{d-q-1} & 0 \\ 0 & I_{q+1} \end{bmatrix},$$

If the rows of M are indexed by λ hyperplanes containing L , $k - \lambda$ hyperplanes which meet L at β_1 , $k - \lambda$ hyperplanes which meet L at β_2 , and so on, then we have

$$MF_1 = \begin{bmatrix} (k - q - 1)\mathbf{1}_\lambda & J_{\lambda \times (q+1)} & 0 \\ (k - 1)\mathbf{1}_{k-\lambda} & \mathbf{1}_{k-\lambda} & 0 \\ \vdots & \ddots & \\ (k - 1)\mathbf{1}_{k-\lambda} & 0 & \mathbf{1}_{k-\lambda} \end{bmatrix}, \quad (5)$$

A spread of $\text{PG}(3, q)$ is a set of lines which partition the set of points. A spread in $\text{PG}(3, q)$ exists for any prime power q . Let $\mathcal{S} = \{L_1, \dots, L_{q^2+1}\}$ be a spread in $\text{PG}(3, q)$. Let M be the plane-point incidence matrix of $\text{PG}(3, q)$, and suppose that the columns of M are indexed in accordance with the partition \mathcal{S} of the points of $\text{PG}(3, q)$. Consider the following $(q^2 + 1)(q + 1) \times (q^2 + 1)$ matrix

$$F_2 = \begin{bmatrix} \mathbf{1}_{q+1} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{q+1} \end{bmatrix}.$$

If the rows of M are indexed by $q + 1$ planes containing L_1 , $q + 1$ planes containing L_2 , and so on, then we have

$$MF_2 = \begin{bmatrix} (q+1)\mathbf{1}_{q+1} & & \mathbf{1}_{q+1} \\ & \ddots & \\ \mathbf{1}_{q+1} & & (q+1)\mathbf{1}_{q+1} \end{bmatrix}. \quad (6)$$

Theorem 4.1. *Let \mathcal{X} be an association scheme of class $d = (q^{m+1} - 1)/(q - 1)$ with the first eigenmatrix*

$$P = \begin{bmatrix} 1 & f\mathbf{1}_d^T \\ \mathbf{1}_d & rM + s(J - M) \end{bmatrix},$$

where M is an incidence matrix of $\text{PG}(m, q)$. Let $k = (q^m - 1)/(q - 1)$. Then the following statements hold.

(i) *There exists a fusion scheme of class $q + 2$ with the first eigenmatrix*

$$\begin{bmatrix} 1 & (d - q - 1)f & f\mathbf{1}_{q+1}^T \\ 1 & (k - q - 1)r + (d - k)s & r\mathbf{1}_{q+1}^T \\ \mathbf{1}_{q+1} & ((k - 1)r + (d - k - q)s)\mathbf{1}_{q+1} & (r - s)I + sJ \end{bmatrix}. \quad (7)$$

(ii) *If $m = 3$, then there exists an amorphous fusion scheme of class $q^2 + 1$ with the first eigenmatrix*

$$P = \begin{bmatrix} 1 & (q + 1)f\mathbf{1}_{q^2+1}^T \\ \mathbf{1}_{q^2+1} & q(r - s)I + (r + sq)J \end{bmatrix}.$$

Proof. (i) We can see easily from (5) that the matrix $(rM + s(J - M))F_1$ has $q + 2$ distinct rows, which are precisely those of the lower-right $(q + 2) \times (q + 2)$ submatrix of (7). Then the result follows from the Bannai–Muzychuk criterion.

(ii) The proof is similar to (i), noting that the matrix $(rM + s(J - M))F_2$ has $q^2 + 1$ distinct rows. \square

Example 1. Let α be an arbitrary primitive element of $\text{GF}(2^{12})$, and let

$$H_j = \{(x, y) \mid x - y \in \alpha^j \langle \alpha^{45} \rangle\} \quad (j \in \mathbb{Z}).$$

For a fixed integer a which is relatively prime to 15, we put

$$R_k = \bigcup_{i=0}^2 H_{a(3(k-1)+5i)}.$$

By computer, we have verified that the graph Γ_k on $\text{GF}(2^{12})$ with edge set R_k is a strongly regular graph with eigenvalues 273, 17, -15 , for each $k \in \{1, \dots, 15\}$. In fact, these graphs are one of the strongly regular graphs discovered by de Lange [14]. Together with the diagonal relation R_0 , we obtain a 15-class pseudocyclic association scheme $(\text{GF}(2^{12}), \{R_i\}_{i=0}^{15})$ satisfying the hypothesis of Theorem 3.1. The principal part of the first eigenmatrix is a linear combination of the all-ones matrix and an incidence matrix of a symmetric 2-(15, 7, 3) design. Since this matrix is circulant by the definition of R_k , the design is isomorphic to $\text{PG}(3, 2)$ by [11, p. 984].

By Theorem 4.1(i), we obtain a 4-class fusion scheme with the first eigenmatrix given by (1). By Theorem 4.1(ii), we obtain a 5-class pseudocyclic amorphous association scheme. We have verified by computer that this amorphous association scheme is not isomorphic to the amorphous cyclotomic association scheme of class 5 on $\text{GF}(2^{12})$.

Example 2. Let α be an arbitrary primitive element of $\text{GF}(2^{20})$, and let

$$H_j = \{(x, y) \mid x - y \in \alpha^j \langle \alpha^{75} \rangle\} \quad (j \in \mathbb{Z}).$$

For a fixed integer a which is relatively prime to 15, we put

$$R_k = \bigcup_{i=0}^4 H_{a(5(k-1)+3i)}.$$

By computer, we have verified that the graph Γ_k on $\text{GF}(2^{20})$ with edge set R_k is a strongly regular graph with eigenvalues 69905, 273, -239 , for each $k \in \{1, \dots, 15\}$. Together with the diagonal relation R_0 , we obtain a 15-class pseudocyclic association scheme $(\text{GF}(2^{20}), \{R_i\}_{i=0}^{15})$ satisfying the hypothesis of Theorem 3.1. The principal part of the first eigenmatrix is a linear combination of the all-ones matrix and an incidence matrix of a 2-(15, 7, 3) design. Since this matrix is circulant by the definition of R_k , the design is isomorphic to $\text{PG}(3, 2)$ by [11, p. 984].

By Theorem 4.1(i), we obtain a 4-class fusion scheme with the first eigenmatrix given by (2). By Theorem 4.1(ii), we obtain a 5-class pseudocyclic amorphous association scheme. We conjecture that the strongly regular graphs in this association scheme are not isomorphic to a cyclotomic strongly regular graph, and hence our amorphous association scheme is new.

Example 3. Let α be an arbitrary primitive element of $\text{GF}(2^{21})$, and let

$$H_j = \{(x, y) \mid x - y \in \alpha^j \langle \alpha^{49} \rangle\} \quad (j \in \mathbb{Z}).$$

For a fixed integer a which is relatively prime to 7, we put

$$R_k = \bigcup_{i=0}^6 H_{a(7(k-1)+i)}.$$

By computer, we have verified that the graph Γ_k on $\text{GF}(2^{21})$ with edge set R_k is a strongly regular graph with eigenvalues 299593, 585, -439 , for each $k \in \{1, \dots, 7\}$. Together with the diagonal relation R_0 , we obtain a 7-class pseudocyclic association scheme $(\text{GF}(2^{21}), \{R_i\}_{i=0}^7)$ satisfying the hypothesis of Theorem 3.1. The principal part of the first eigenmatrix is a linear combination of an incidence matrix of $\text{PG}(2, 2)$ and the all-ones matrix.

By Theorem 4.1(i), we obtain a non-amorphous 4-class fusion scheme of the cyclotomic scheme of class 49 on $\text{GF}(2^{21})$ with the following first eigenmatrix:

$$\begin{bmatrix} 1 & 1198372 & 299593 & 299593 & 299593 \\ 1 & -1756 & 585 & 585 & 585 \\ 1 & 292 & -439 & -439 & 585 \\ 1 & 292 & -439 & 585 & -439 \\ 1 & 292 & 585 & -439 & -439 \end{bmatrix}. \quad (8)$$

This gives the third counterexample to A.V. Ivanov's conjecture having class 4.

Acknowledgement. The authors thank Qing Xiang for bringing the reference [19] to the authors' attention.

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